An anti-classification theorem for minimal homeomorphisms on the torus

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DDC, 2025.9.30

A classification?

Classification problems are of the following form:

Given an analytic equivalence relation E on a standard Borel space X, determine whether two point $x, y \in X$ are equivalent.

Examples

The conjugacy of complex matrices can be classified by the eigenvalues of the matrix.

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Definition

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$$aEb \iff f(a)Ff(b)$$

If a Borel reduction from E to F exists, we would say E is **Borel reducible** to F and denote by $E \leq_B F$. If we also have $F \leq_B E$ then we use the notation $E \sim_B F$, in this way, classifying E is as complicated as classifying F.

A classification?

We want to study equivalence relation E. F is a "well-studied" equivalence relation. We reduce E to F.

An anti-classification?

We want to study equivalence relation F. E is an "impossible" equivalence relation. We reduce E to F.

Impossibility?

We need some benchmarks to measure the impossibility of a problem.

Numerical invariants

An equivalence relation E is called **smooth** or is classifiable by numerical invariants, if it is Borel reducible to $=_{\mathbb{R}}$ where $=_{\mathbb{R}}$ denotes the equality relation on $\mathbb{R}$.

Countable equivalence relation

A Borel equivalence relation is **countable** if every equivalent class is countable.

E_0 is an equivalence relation defined on 2^ω as follows:

$$xE_0y \text{ if } \exists n \forall m \geq n \ x(m) = y(m)$$

Harrington-Kechris-Louveau Theorem

Let E be a Borel equivalence relation, then either E is smooth or E_0 is continuously reducible to E.

Borel equivalence relations

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If an equivalence relation is not Borel, then we can not describe this classification by using inherently **countable** information.

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If an equivalence relation is not classifiable by countable structures, it is impossible to classify it by any algebraic invariants.

Maximal equivalence relations

A **complete** element in a partially ordered class is the most complicated element in that class.
For Polish group actions, S_∞ actions, countable Borel equivalence relations, complete elements exists.

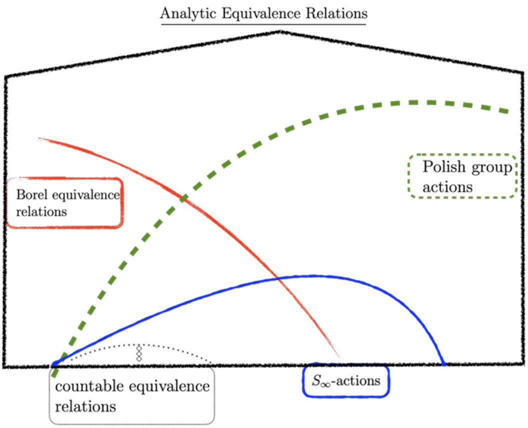

Figure 2: Basic regions of complexity

Figure: A picture from Foreman

Dynamical system

We care about the following two types of systems:

1. (X, μ, T) where (X, μ) is a standard probability space and $T \in \mathrm{MPT}(X, \mu)$.
2. (X, f) where X is a compact metric space and $f \in \mathrm{Homeo}(X)$.

Statistical behavior

Figure: von Neumann

Statistical behavior

von Neumann suggested classifying dynamical systems by their **statistical behaviors**.

Conjugacy of MPTs

Definition

Two measure-preserving transformations(MPTs) T, S are conjugate if there exists another MPT, H, such that $HTH^{-1} = S$.

What is preserved?

Integral, ergodicity, . . .

Qualitative behavior

Figure: Smale

Qualitative behavior

Smale suggested classifying dynamical systems by their **qualitative behaviors**.

Topological conjugacy

Definition

Two systems (X, f) and (Y, g) are **topological conjugacy** if there exists a homeomorphism $h : X \to Y$ such that

$$h \circ f = g \circ h.$$

What is preserved?

Fixed points, asymptotic pairs, affine structure of invariant measures, ...

Definition

Let $T \in \mathrm{MPT}(X, \mu)$, T is ergodic if every T-invariant subset of X has measure either 1 or 0.

Definition

A system (X, f) is called minimal if there is no proper subsystems of (X, f). It is equivalent with the condition that all orbits are dense.

Why ergodicity?

Ergodic decomposition theorem

Every measure-preserving transformation could be written as an integral of ergodic measure preserving transformations

Why minimality?

Existence of minimal set

Every topological dynamical system has a minimal subsystem.

!

Ergodic transformations and minimal systems are building blocks of general systems.

Two programs regarding those classifications

The isomorphism problem (von-Neumann, 1936)

Classify MPTs up to conjugacy.

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Smale's program

Classify smooth and topological dynamical systems up to topological conjugacy.

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Theorem (Von-Neumann)

Two MPTs with discrete spectrum are conjugate if and only if their associated Koopman operators have the same eigenvalue. (Reducible to $=^{+}_{\mathbb{R}}$).

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Theorem (Foreman, Rudolph and Weiss, 2011)

The conjugacy relation of ergodic MPTs is not Borel.

Theorem (Foreman and Weiss, 2021)

Conjugacy of measure preserving diffeomorphisms on the 2-torus is not Borel.

Theorem (Gerber and Kunde, 2025)

Kakutani equivalence of ergodic transformations is not Borel.

Theorem (Foreman, 2025+)

Isomorphism of countable graphs is Borel reducible to conjugacy of ergodic diffeomorphisms on the 2-torus.

Cantor minimal systems

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Topological full groups

Let $(\mathcal{C}, f)$ be a Cantor system. Topological full group $[f]$ is a countable group determined only by f. The map sends f to $[f]$ is continuous.

Flip conjugacy

Two systems (X, f) and (Y, g) are **flip conjugate** if (X, f) is conjugate with (Y, g) or (Y, g^{-1}).

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We find an algebraic invariant!

Theorem (Carmelo, Gao, 2001)

The conjugacy relation of Cantor systems is a complete S_∞ action.

Theorem (Deka, García-Ramos, Kasperzak, Kunde, Kwietniak, 2024+)

The conjugacy relation of Cantor minimal systems is not Borel.

Rotation number

Let f be a minimal homeomorphism on the circle S^1. Let F be the lift of f on $\mathbb{R}$.

Theorem (Poincaré, 1907)

1. The limit of $\frac{F^n(x)-x}{n}$ exists and independent of the choice of $x \in \mathbb{R}$. We call this number the **rotation number** of f.
2. Two minimal homeomorphisms on the circle are conjugate if and only if they have the same rotation number.
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There is a numerical invariant.

In general?

Theorem (Foreman and Gorodetski, 2022)

Let M be a manifold with dimension n, then the topological conjugacy relation of smooth diffeomorphisms on M is

1. not smooth if $n \geq 2$.
2. not Borel if $n \geq 5$.

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Foreman and Gorodetski, Vejnar independently generalized non-Borelness to all manifolds.

Algebraic invariants?

Question (Foreman and Gorodetski)

Does topological conjugacy of diffeomorphisms on a given manifold reduce to an S_∞ action?

Theorem(P. 2025)

For any manifold M with dimension greater than equal to 2, the topological conjugacy of diffeomorphisms on M is not classifiable by countable structures.

Hjorth's result

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Theorem (Hjorth,1999)

The conjugacy of homeomorphisms on the square is not classifiable by countable structures.

Generalize Poincaré's theorem?

a torus? It seems natural to attempt to generalise Poincaré's result to higher dimensions. However, so far no results in this direction exist. Partly, this is

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A natural question:

Can we prove any (anti)classification results for minimal homeomorphisms on the torus?

Foreman's question:

Open Problem 14. *Does E_0 reduce to the collection of topologically minimal diffeomorphisms of the 2-torus with the relation of topological conjugacy? What about topologically transitive diffeomorphisms?*

Theorem (P. 2025+)

E_0 is Borel reducible to the topological conjugacy of minimal diffeomorphisms on the torus.

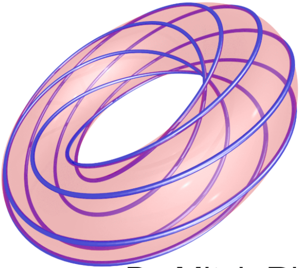
Irrational rotations

Let $\alpha, \beta \in \mathbb{T}$ which are rationally independent. Define

$$T_{\alpha,\beta} : \mathbb{T}^2 \to \mathbb{T}^2$$

$$T_{\alpha,\beta}(x,y) = (x+\alpha, y+\beta).$$

Then $T_{\alpha,\beta}$ is minimal.



By Mitch Richling

Proof is not long!

A fact from dynamical system

Let $(\alpha,\beta),(\alpha',\beta')\in\mathbb{T}^2$, two minimal rotations $T_{\alpha,\beta},T_{\alpha',\beta'}$ are conjugate iff $\exists A\in \mathrm{GL}_2(\mathbb{Z})$ such that $A(\alpha,\beta)=(\alpha',\beta')$.

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Proof

Let h be a conjugacy between two rotations. We may assume $h(0,0) = (0,0)$, thus h is a group isomorphism between $(n\alpha, n\beta)$ and $(n\alpha', n\beta')$. Since the orbit is dense, we know h is a self-isomorphism on $\mathbb{T}^2$. Thus, the lift of h on the plane, H, is also a self-isomorphism, since H is a lift, H maps $\mathbb{Z}^2$ to $\mathbb{Z}^2$. Thus, $H \in \mathrm{GL}_2(\mathbb{Z})$.

A fact from group theory

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A fact from measure theory

The action of $\mathrm{GL}_2(\mathbb{Z})$ on the 2-torus preserves Lebesgue measure.

A classification in general?

Hjorth proved the conjugacy relation of $\mathrm{Homeo}([0,1]^2)$ is not classifiable by countable structures. But Hjorth's proof uses fixed points in an essential way.

Theorem (P. 2025+)

The topological conjugacy relation of minimal homeomorphisms on 2-torus is not classifiable by countable structures.

AbC method

The contribution of Anosov and Katok

The **approximation by conjugation(AbC)** method was invented by Anosov and Katok in 1970s to construct new dynamical systems.

How it works on the 2-torus?

Let R be a minimal rotation on the 2-torus and h_n be a sequence of homeomorphisms on the 2-torus. Take the limit of $h_n R h_n^{-1}$.

Asymptotic pairs

In a topological dynamical system (X, d, f), two points $x, y \in X$ are **asymptotic** if the limit of $d(f^n x, f^n y)$ goes to 0.

!

This is an equivalence relation! And it is preserved under conjugacy!

No fixed points but...

!

Take $X = \mathbb{T}^2$. By adding conditions to h_n, for all $x \in \mathbb{T}^2$, the number of elements in the asymptotic class of x is either finite or continuum.

!!

The elements with continuum asymptotic class must be mapped to elements with continuum asymptotic class.

!!!

Those points can play the same role as fixed points.

Open Question

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All Choquet simplices can be realized as the set of invariant measures of a Lebesgue measure preserving diffeomorphism on the 2-torus.

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Question

Are there any relations between those two theorems?

Thanks.